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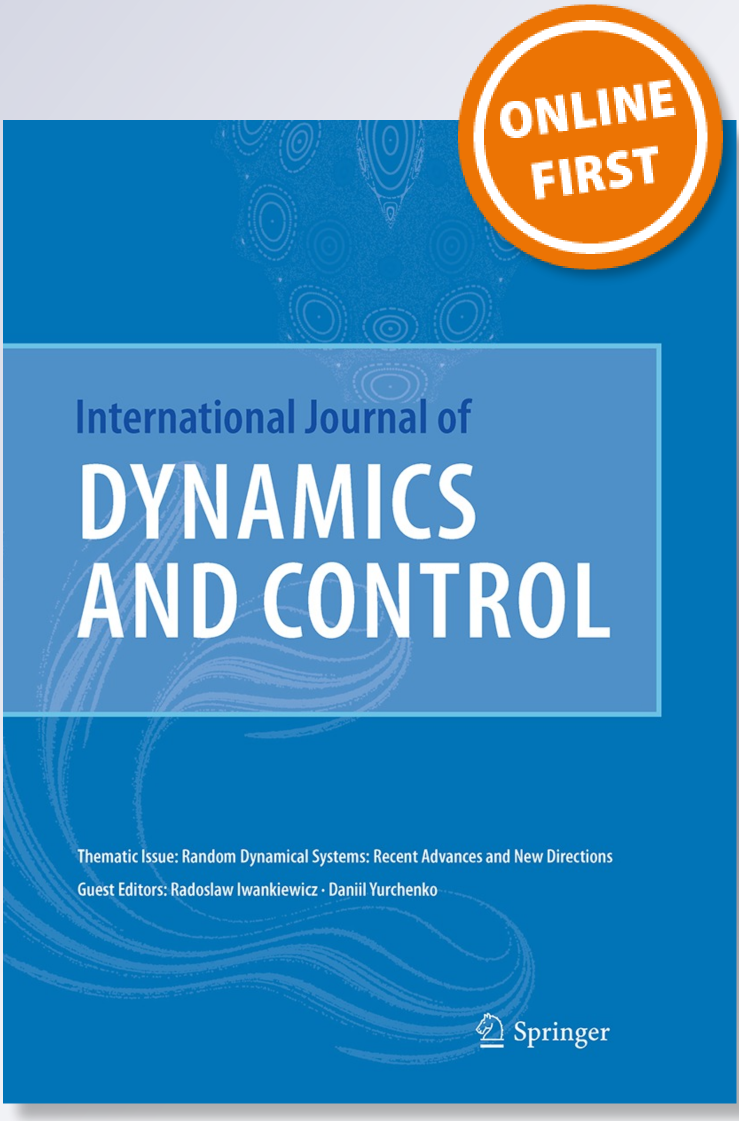


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# Dynamics of a parametric rotating pendulum under a realistic wave profile

**Tatiana Andreeva[1] · Panagiotis Alevras[1] · Arvid Naess[2] · Daniil Yurchenko[1]**

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**Abstract** Stochastic dynamics of a rotating pendulum is investigated upon the development of a novel wave energy converter. A dependency of the pendulum's rotational potential on the wave profile is studied. The latter is modeled using a non-harmonic periodic function with a sharp or bent crest. It has been shown that the sharp wave profile significantly deteriorates the rotational potential of the parametric pendulum, so that the system becomes non-rotational even in the primary parametric resonance zone.

**Keywords** Parametric pendulum · Parametric resonance · Random phase modulations · Narrow band process · Wave energy · heaving motion · Wave profile

## 1 Introduction

The parametric excitation phenomenon and its applications have attracted significant attention from researchers all over the world, especially in the last decade. In the case of a parametrically excited pendulum, this phenomenon leads to rotational motion of the pendulum when its suspension point is excited along a line or in a plane. Thus it may be thought of as a mechanism converting one type of motion (rectilinear for instance) to another type of motion (rotational). This is especially important since a conventional way of generating electricity is through rotational motion of a rotor. Based on that, a concept for a wave energy converter has been proposed where the heaving motion of ocean waves provide a vertical excitation to a pendulum's pivot, aiming at rotational response [1].

In-plane dynamics of a parametrically excited pendulum is well described by the Mathieu equation. It is a well-established fact, that systems described by the Mathieu equation have domains of stable and unstable motion. The primary parametric resonance occurs when the excitation frequency is double the natural frequency of the system. In the absence of friction, the instability of the fixed point at the primary resonance frequency can happen at very small values of the excitation amplitude. Therefore it seems reasonable to expect a pendulum's rotational response at the primary parametric resonance.

Until recently, the majority of research papers were not interested in the energy harvesting potential of the parametric pendulum and therefore were focused on the deterministic dynamics of the parametrically excited pendulum, including chaotic dynamics, bifurcation analysis, as well as a parametric response of a double pendulum and synchronization of pendulums (see [2–9] and references therein). However, the application of the parametric pendulum as a wave energy harvesting device opened another research direction related to the stochastic nature of waves. A number of papers has been published recently and devoted to the the rotational motion in the random sea environment [10–14] and references therein. Although the high noise intensity makes it difficult to achieve a sterling rotational response, the reported results have demonstrated that it was possible to achieve sustainable rotations under a stochastic sea-like excitation. The latter was modeled using a harmonic function with a random phase modulation technique.

Since real sea waves may not have a smooth harmonic profile, it is of interest to understand the influence of the waves profile on the rotational pendulum response. Thus,

✉ Daniil Yurchenko
d.yurchenko@hw.ac.uk

1 Institute of Mechanical, Process and Energy Engineering, Heriot-Watt University, Edinburgh EH14 4AS, UK

2 Department of Mathematical Sciences, NTNU, Trondheim, Norway

in this paper the authors adopt a more realistic wave profile with a sharp crest peak, which resembles in shape, when deterministic, a cycloid turned upside down. A number of numerical simulations, deterministic and stochastic are conducted to investigate the influence of the wave profile onto the rotational potential of the parametric pendulum.

## 2 Wave profile modeling

Looking at sea waves it is easy to observe that the waves' shape is hardly sinusoidal and has a sharp crest profile, which is especially noticeable at shallow waters. Moreover, the waves crest may become asymmetric before breaking. To better model such a wave profile a generalized cycloid curve will be used. If one fixes a point on a wheel rim and let the wheel roll along a straight line, the point will describe a curve known as cycloid. This curve has a number of features, one of which is a sharp angle of the curve when the point touches the line. The following equation:

$$\begin{aligned} x &= -\frac{p}{k} - r sin(p) \\ y &= -rk cos(p-\gamma) \end{aligned} \tag{1}$$

where $p$—parameter, $k$ and $r$—two constants ($r$ is the wheel radius), $\gamma$—phase angle, describes a more general curve. When $\gamma = 0$ and $kr$ is less than unity, the wave shape tends to be harmonic, which is presented in Fig. 1 for $kr = 0.5$ with $k = 0.5, r = 1.0$ on the top and $k = 1.0, r = 0.5$ on the bottom. Based on these figures one can define $k$ as the frequency of the profile. It can also be seen that the profile is asymmetric with respect to the horizontal axis in the sense that the amount of time spent under the horizontal line within a period is bigger than that above the abscissa. Note that the increase in the asymmetry is associated with sharper wave crest, so that for $rk = 1$ the positive part of the profile takes only 18 % of the period. On the opposite, for very small values of $kr \ll 1$, the profile resembles a purely symmetrical harmonic function with 50–50 spent above and below abscissa.

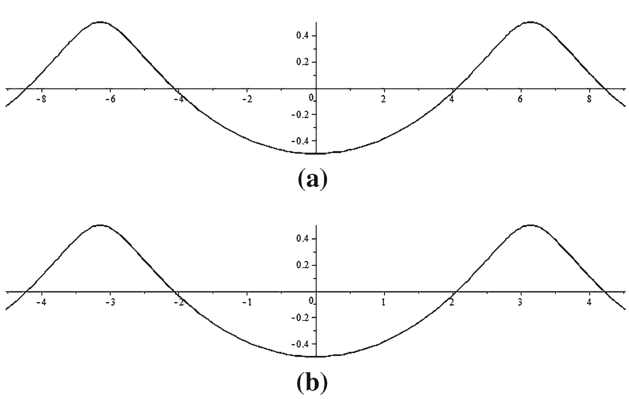


**Fig. 1** Profile from (1) for $\gamma = 0$ and **a** $k = 0.5, r = 1.0$, **b** $k = 1.0, r = 0.5$

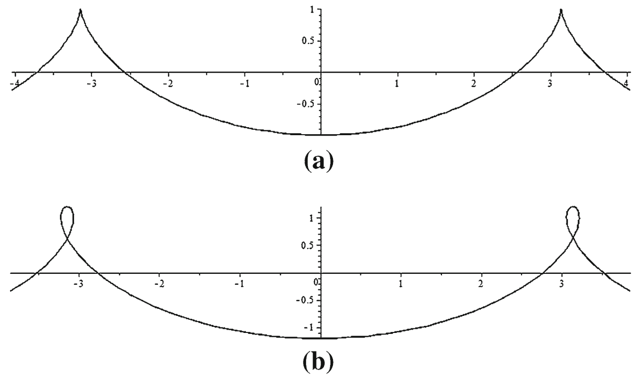


**Fig. 2** Profile from (1) for $\gamma = 0$ and **a** $kr = 1.0$, **b** $kr = 1.2$

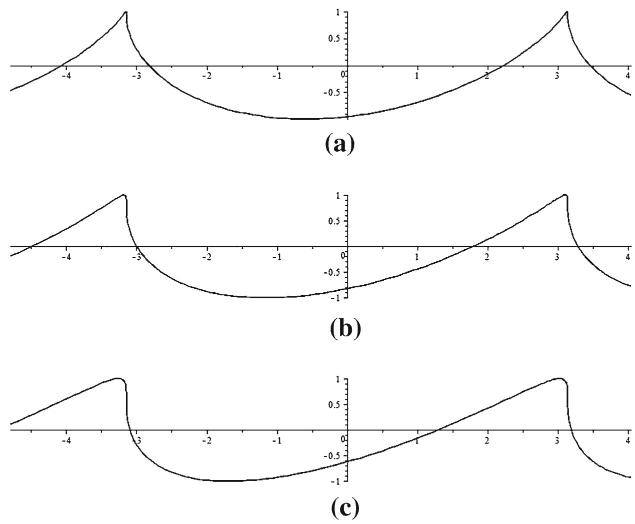


**Fig. 3** Profile from (1) for $kr = 1.0$ and **a** $\gamma = 0.3$ **b** $\gamma = 0.6$, **c** $\gamma = 0.9$

For $\gamma = 0$ and value of $kr = 1$ Eq. (1) defines a classical cycloid, whereas for values $kr > 1$ the curve will have self intersections or loops, which are clearly shown in Fig. 2 for $kr = 1$ (top) and $kr = 1.2$ (bottom).

When $\gamma \neq 0$ the wave profile near the crest has a tendency to bend, which is demonstrated in Fig. 3. It is seen that increasing the phase angle $\gamma$ leads to a smother wave crest and larger percentage of the positive part of the wave, reducing the asymmetry. These wave profiles are used later as a parametric excitation in order to understand the influence of the former onto the rotational potential of the parametric pendulum.

## 3 Deterministic dynamics of a parametric pendulum

Deterministic dynamics of a pendulum with a vertically moving suspension point may be expressed as:

$$\ddot{\theta} + 2\alpha\dot{\theta} + [1 + f(\nu t)/g]\sin\theta = 0$$
$$\lambda = \frac{A\omega^2}{g} \tag{2}$$

where $A$ and $\omega$ are the wave amplitude and frequency, $g$—the acceleration of gravity, $\nu$ is the ratio of the excitation and natural frequencies. A great number of papers were devoted to the analysis of Eq. (2) for linear ($sin(\theta) \approx \theta$) and nonlinear versions of (2), as well as various types of $f(t)$ function—harmonic, periodic or random. In [15] the response of a parametric pendulum to the square sinusoidal excitation has been studied.

However, to the best knowledge of the authors, the influence of a realistic waves profile onto the dynamics of the parametric pendulum has never been investigated. The results will be presented in the parameter space ($\lambda$, $\nu$) in order to be consistent with results presented earlier by the authors. In the present modeling $f()$ function is taken in the form of Eq. (1). Since equation (1) can only be written in the parametric form it creates a certain difficulties in adapting this equation to be used in Eq. 2 instead of $f()$. Therefore, a wave profile was generated first in a form of two arrays $x$, $y$, then the first array $x$ became a time variable, whereas the second one became the excitation function $f()$.

To make sure the presented results are correct, first a test was carried out with $kr = 0.01$, which is shown in Fig. 4. These results match identically the results reported earlier by the authors [14,16], which validates the code. The red area indicates the region of dominant rotational motion (over 90 %) whereas the blue area indicates dominant oscillatory motion or no motion at all (fixed point). There is no differentiation between the clockwise and counter clockwise rotations, as well as rotations with different periods since it is not of major importance from the electricity generation point of view.

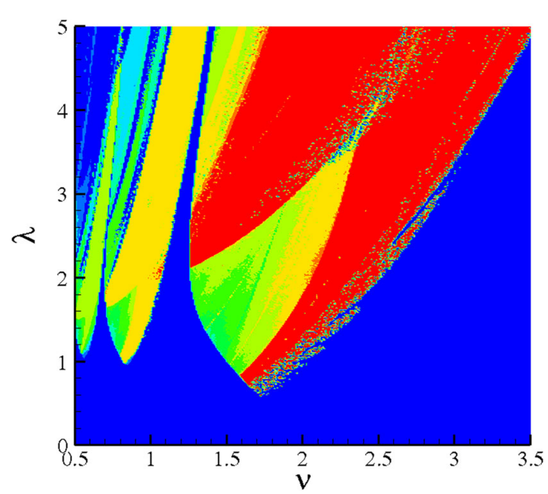

Fig. 4 Deterministic map for $\gamma = 0$ and $kr = 0.01$

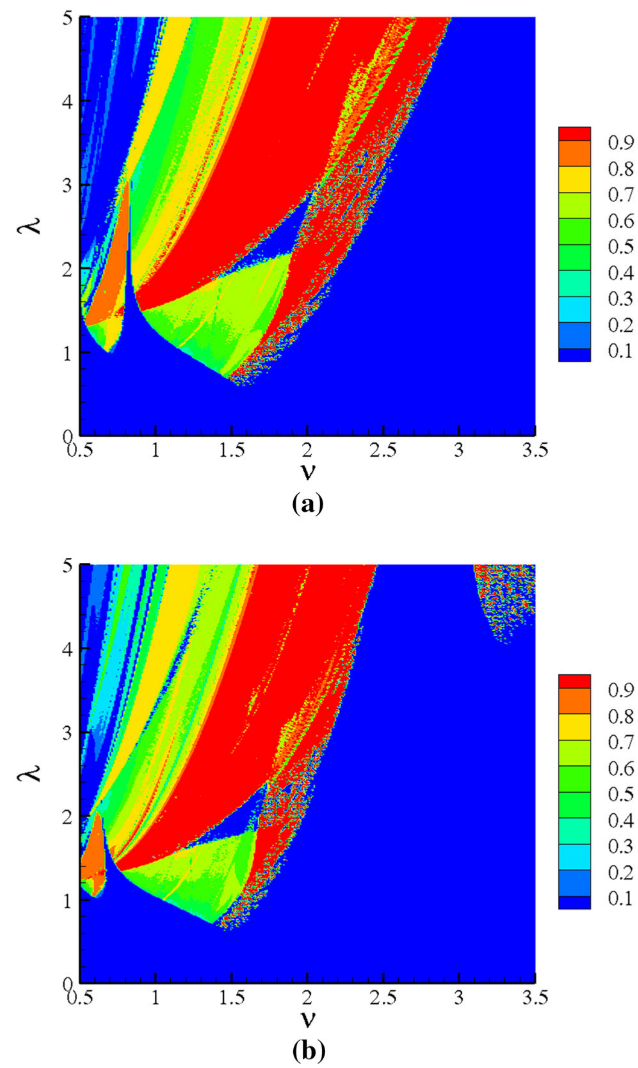

Fig. 5 Deterministic map for $\gamma = 0$ and **a** $kr = 0.5$, **b** $kr = 0.75$

Next, two plots in Fig. 5 demonstrate the influence of the smooth profile with bigger asymmetry with respect to the horizontal axis and sharper wave crest. One can observe the difference between Figs. 4 and 5a, noting the blue domain in the middle and significant change in the behavior at low values of $\nu$. Moreover it seems like the entire picture has been shifted slightly to the left. The latter observation is confirmed on Fig. 5b where one clearly sees the appearance of higher order instability domain in the top right corner. The "granular" structure of the red domain in this corner, as well as at values of $\nu \in [1.5, 2.0]$ indicates the chaotic character of the response. These observations may be considered as some negative effects of the excitation asymmetry onto the rotational potential of the parametric pendulum, since the goal is a sustainable rotational motion, which is mostly observed in the red domains.

Coming to the model of the perfectly sharp wave crest ($rk = 1$), depicted on Fig. 6, it should be especially noted

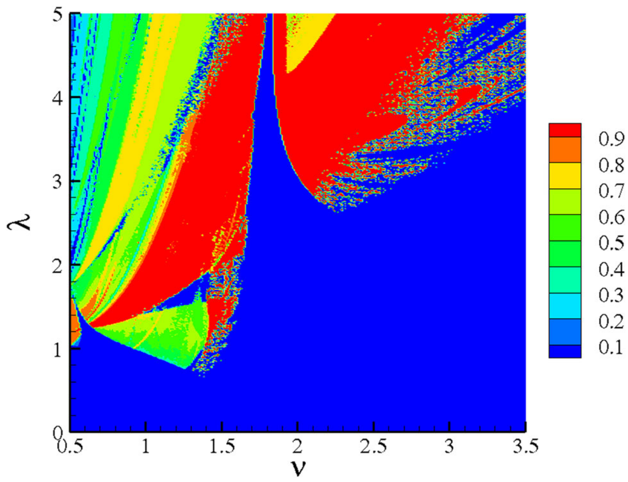


**Fig. 6** Deterministic map for $\gamma = 0$ and $kr = 1$

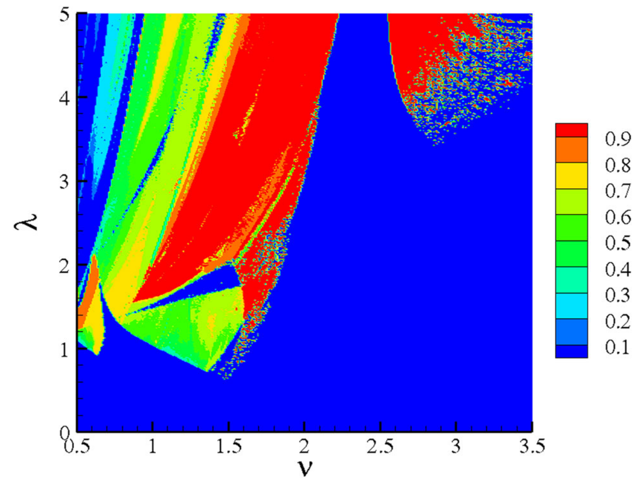


**Fig. 8** Deterministic map for $\gamma = 0.6$ and $kr = 1$

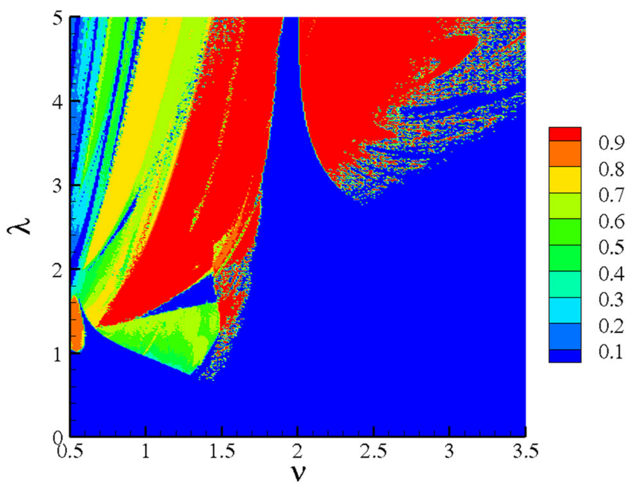


**Fig. 7** Deterministic map for $\gamma = 0.3$ and $kr = 1$

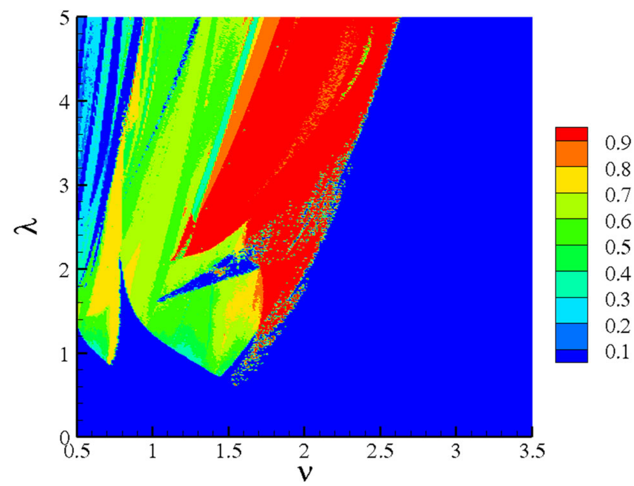


**Fig. 9** Deterministic map for $\gamma = 0.9$ and $kr = 1$

right away that at values around the primary parametric resonance $\nu = 2$ the system has lowest rotational potential. This is a quite surprising and definitely unexpected result. Note that the blue "tongue" is to the left from $\nu = 2$, so it is expected that for values of $kr$ close to unity from the left it will be exactly at the primary parametric resonance value.

In the case of a harmonic excitation, deterministic or random, the right boundary of the rotational domain has always remained more or less unchanged, resembling the one on Fig. 4. In Figs. 5 and 6 one can observe significant reduction of the rotational domain from the right side. Similar trend can be seen in the subsequent figures for "bent" wave crest. Figure 7 presents results of the numerical simulation for $\gamma = 0.3$ and $kr = 1$, which correspond to the case of the sharpest bent wave from Fig. 3a. This map looks very much like the previous ones, with a slight increase of the bottom green area to the right. With increase of $\gamma$, the crest sharpness decreases and the map tends to look like one depicted earlier for $kr < 1$. In Figs. 8 and 9 one can see the maps for values of $\gamma = 0.6$ and $\gamma = 0.9$ correspondingly. It can be seen that the entire map moves back to the right, the rotational domain in the right top corner has been shifted out from the map and the bottom green area has been increased. However, the right boundary has not been returned to its original state corresponding to a harmonic excitation or very small values of $rk$ (see Fig. 4). These maps and their described behavior indicate that the system's rotational potential depends on the sharpness of the peak, which is in this model directly related to the asymmetry of the excitation, rather than the crest bend. In fact the latter, connected to the profile asymmetry, influences the map through it.

## 4 Stochastic dynamics of the rotating pendulum

To understand the pendulum response to waves with a realistic profile and spectra, a random phase modulations technique of the waves is used. The pendulum's equation of motion in this case will be similar to Eq. (2) with a small difference:

$$\begin{aligned} &\ddot{\theta} + 2\alpha\dot{\theta} + [1 + f(q)/g]\sin\theta = 0 \\ &\dot{q} = \nu + \sigma\xi(t), \end{aligned} \tag{3}$$

where $\nu$—mean frequency, $\sigma^2 = D$ is noise intensity and $\xi(t)$—a delta-correlated Gaussian white noise with zero mean. Varying $D$, it is possible to simulate various wave spectra, however it should be mentioned that only a central part of the spectrum is well approximated due to the model simplicity. In particular, the model (3) may be used to approximate Pierson-Moskowitz spectra with $D = 0.3$ [13].

It was reported for the deterministic system that the most critical case happens when $kr = 1$ and $\gamma = 0$, thus it seems reasonable to study the system response with this particular wave profile and random phase modulations for various values of the noise intensity in order to study its influence onto the pendulum rotational potential. Figures 10 and 11 demonstrates the results of numerical simulations for $D = 0.1$ and $D = 0.3$ correspondingly. If one compares Figs. 10 with 6 it can be seen that the noise has smoothened the picture overall, removing the stable "bay" at the primary resonance zone, which may be considered as a benefit for the pendulum response. On the other hand, it has reduced the maximum rotational percentage to 80 % and shifted this domain up to higher values of $\lambda$. It should also be noted that the zone of the maximum rotational potential is finite and constrained from an upper value of $\lambda$, which is different from previously reported results for a harmonic (symmetric) wave profile with random phase modulations, where the dominant rotational domain is unbounded from the above. Increasing the noise intensity leads to further reduction of the pendulum's rotational potential, which can be observed in Fig. 11.

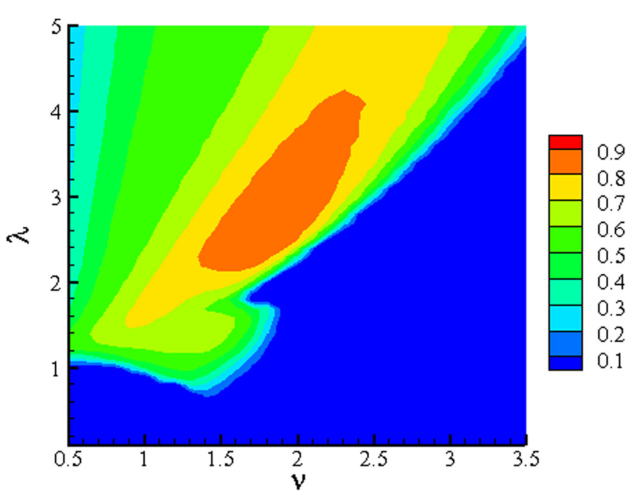


**Fig. 10** Stochastic map for $D = 0.1$, $\gamma = 0.0$ and $kr = 1$

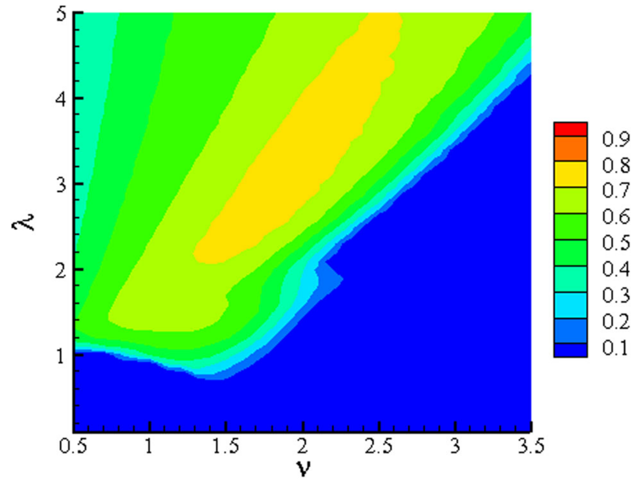


**Fig. 11** Stochastic map for $D = 0.3$, $\gamma = 0.0$ and $kr = 1$

## 5 Conclusions

This paper considers the influence of a realistic wave profile onto the parametric pendulum's rotational potential. The former is modeled using a formula of a generalized cycloid, so that the wave has either a sharp or bent crest. The considered model provides a non-symmetric, with respect to a horizontal axis, wave profile, increasing the lower portion of the wave with increased sharpness. The pendulum is modeled as a nonlinear single-degree-of-freedom system with its pivot point excited in the vertical direction due to the heaving motion of waves. It has been demonstrated that the wave asymmetry influences significantly the rotational potential of the parametric pendulum, not only by substantially changing the parameter space map of the response, but also making it non-rotational at the primary resonance zone. Moreover, it has been demonstrated that the crest bend does influence the rotational map through the asymmetry of the wave profile, which is reduced with increasing phase difference.

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